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April 1973

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Survey Review XXII, 168, April 1973

A CONFORMAL MAPPING PROJECTION WITH MINIMUM SCALE ERROR

W. I. Reilly

Geophysics Division, Department of Scientific and Industrial Research, Wellington, New Zealand

ABSTRACT

A conformal (orthomorphic) mapping projection of the spheroid can be constructed to give minimum scale error over a given arbitrary area, and in this respect has an advantage over more regular projections such as the transverse Mercator or the Lambert conformal conic. Geodetic coordinates on the spheroid are first transformed into isometric coordinates, and the latter are then transformed into the rectangular cartesian coordinates of the desired projection by means of a polynomial expression in complex variables. The total distortion of the projection is expressed as the integral of the squared scale error over the given area. After fixing the values of the rectangular coordinates and of the meridian convergence at the origin of the projection, the remaining coefficients of the complex polynomial are adjusted to minimise the total distortion. This set of coefficients can be used directly in formulae to carry out the direct and inverse transformations between geodetic and rectangular coordinates, and to calculate the scale factor, the meridian convergence, and the geodesic curvatures of projected curves (including meridians and parallels) at any point. In the reduction of the observations of local surveys in rectangular coordinates, the minimum scale error property means that corrections to bearings and distances are often negligible, or if required they can be interpolated from small-scale contour maps.

As an example, coefficients have been calculated for a projection designed to give minimum distortion over the land area of New Zealand, using a complex polynomial to order six. The range of scale error for this projection is about 4×10^{-4}, less than can be obtained with any conventional projection.

INTRODUCTION

The representation of the curved surface of the earth spheroid on a flat sheet of paper can be achieved by any one of a large number of mapping projections, which have an extensive history and have been the subject of an equally extensive literature. A mapping projection may be specified by a mathematical transformation whereby the geodetic coordinates ϕ (latitude) and λ (longitude) of a set of points on the spheroid are uniquely transformed into a set of rectangular coordinates x (north) and y (east) on the mapping plane. In this paper we shall be concerned with one class of conformal (orthomorphic) projections which are suitable for medium to large scale topographic maps (1 : 250,000 or greater), and which from their minimum scale error property, are also suitable for the reduction of the observations of local surveys directly in terms of the rectangular (x, y) coordinates with small and often negligible corrections to distances and bearings.

Conformal projections are widely used for these purposes, particularly in the form of the transverse Mercator (Gauss–Krüger) and Lambert conic projections; a comprehensive account of the regular conformal projections of the spheroid has been given by Hotine [**4**]. In New Zealand, the transverse Mercator projection (Lee, [**6**]) is used for topographic maps with a separate projection for each of the North and South Islands (Fig. 1), and the associated rectangular coordinates in yards (the National Yard Grid) are printed on the maps and used extensively as a convenient coordinate reference system for many purposes. Local survey computations, however, are done in transverse Mercator coordinates in twenty eight separate cadastral coordinate regions.

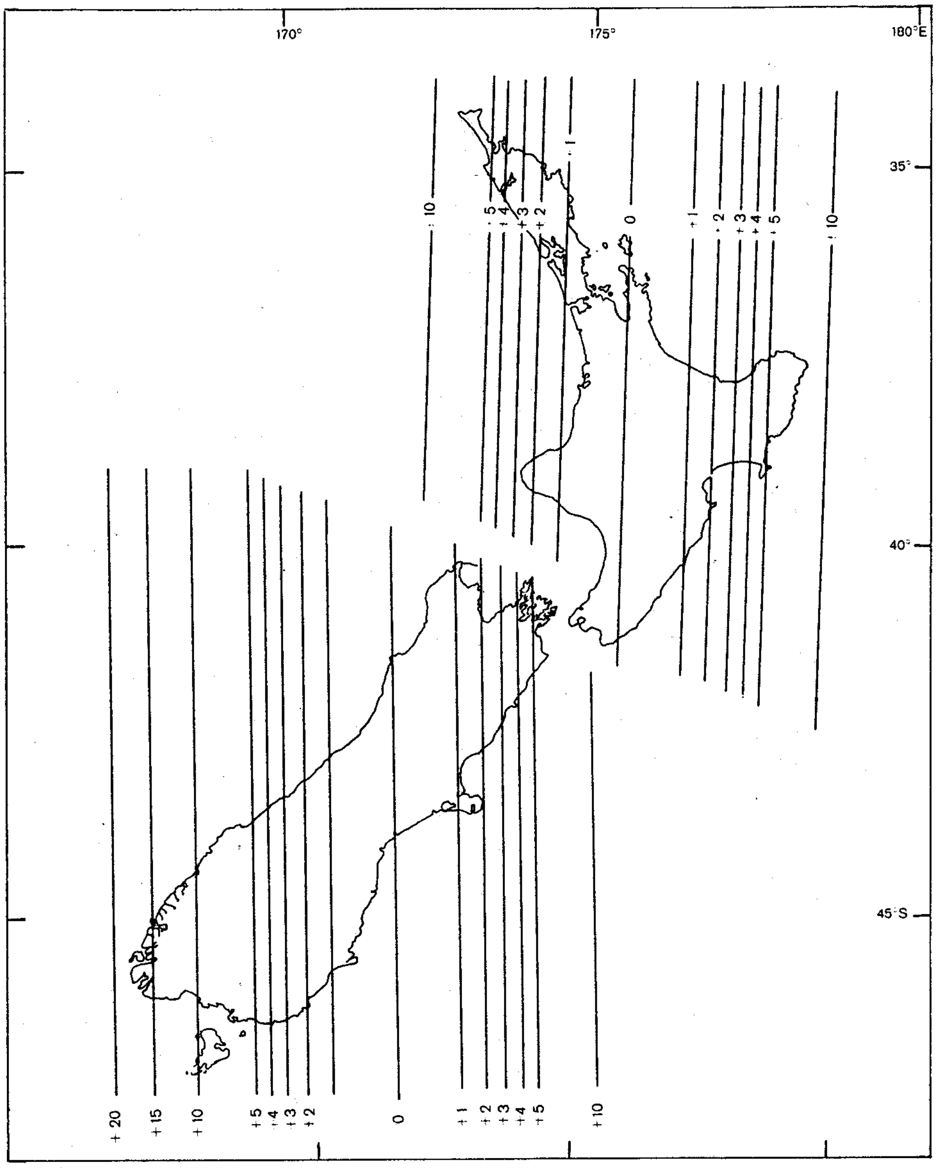

Fig. 1. Curves of equal scale error of the two existing transverse Mercator projections for New Zealand. Scale error in units of 10^{-4} (100 mm/km).

The impending change to metric units of measurement in surveying and mapping in New Zealand has prompted the search for a single new projection and coordinate system which might replace the thirty existing topographic and cadastral coordinate systems. A suitable projection would need to have a range of scale error markedly less than the value of 20×10^{-4} attained by the Transverse Mercator projection of the South Island. To obtain such a conformal projection of small scale error, Mr L. P. Lee, Chief Computer, Lands and Survey Department, suggested that it be designed by first specifying the shape of a regular curve of maximum scale error enclosing the area to be mapped, as was done by Miller [7] for a conformal projection of Europe and Africa. The present investigation is based on an alternative criterion, that of minimising the scale error over the area to be mapped without prior specification of the shapes of the curves of equal scale enlargement.

The history of minimum error projections (Lee, [5], Hendrikz, [3]) appears to go back to Sir George Airy [1], whose concept of minimising the total distortion of linear scale in a map, though originally applied to finding a compromise between conformal and equal-area projections, can be applied to a conformal projection alone. To achieve the greatest minimisation of scale error, it is necessary to abandon the concept of mapping on a regular developable surface, the most general of which is a cone. An oblique conformal conic projection, for example, has at most only four disposable parameters to be assigned to satisfy a scale condition, whereas it is possible to develop conformal transformations in series form, as suggested by Miller [7], in which there is a theoretically unlimited number of parameters which can be adjusted to achieve minimum scale error over an arbitrary area. In practice, of course, only a limited number of coefficients is required to attain an acceptably small scale error, and both the calculation of the coefficients of the projection, and the transformation of coordinates from geodetic to rectangular and vice versa, can be easily done by electronic computer.

Conformal Projection of the Spheroid

In any mapping projection of the spheroid, the length ds of a line element on the spheroid is related to the length $d\bar{s}$ of the corresponding line element on the mapping plane by

$$d\bar{s}^2 = m^2\, ds^2 \tag{1}$$

where m is the linear scale factor. In a conformal projection, m is a continuous differentiable scalar function of position

$$m = m(\phi, \lambda) = m(x, y)$$

where ϕ, λ are the geodetic latitude and longitude on the spheroid, and x, y the rectangular north and east coordinates on the mapping plane. The scale error of a conformal projection can be defined as

$$k = m - 1 \tag{2}$$

In general, if u, v are a pair of orthogonal isometric coordinates on one surface, such that

$$ds^2 = \mu^2(du^2 + dv^2) \tag{3}$$

and $\bar{u}$, $\bar{v}$ the corresponding pair of coordinates on a second surface, then it can be shown (e.g. Näbauer, [8]) that a conformal transformation can be expressed as

$$\bar{w} = f(w) \tag{4}$$

where $\bar{w} = (\bar{u}, \bar{v})$ and $w = (u, v)$ are complex quantities*, and f is a continuous analytic function. The scale factor m is given by

$$m = \frac{\bar{\mu}}{\mu} \left| \frac{d\bar{w}}{dw} \right| \tag{5}$$

* See Appendix for complex number notation

If the first surface is the spheroid, then

$$ds^2 = (\rho\, d\phi)^2 + (p\, d\lambda)^2 \tag{6}$$

where

$$\rho = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}$$

is the radius of curvature of the meridian, and

$$p = \frac{a\cos\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}$$

is the radius of the parallel of latitude, a and b being the equatorial and polar radii of the spheroid, and e the eccentricity given by $e = (a^2-b^2)^{\frac{1}{2}}/a$. The isometric coördinates on the spheroid are taken as the isometric latitude ψ and the longitude ω measured from some convenient origin ϕ_0, λ_0, viz.

$$\psi = \int_{\phi_0}^{\phi} (\rho/p)\, d\phi \tag{7}$$

$$\omega = \lambda - \lambda_0 \tag{8}$$

The isometric latitude is defined so that

$$d\psi = (\rho/p)\, d\phi \tag{9}$$

and substituting for $d\phi$ and $d\lambda$ in (6) we have

$$ds^2 = p^2(d\psi^2 + d\omega^2) \tag{10}$$

whence $\mu = p$.

The corresponding isometric coordinates on the mapping plane are the rectangular cartesian coordinates x, y, for which

$$d\bar{s}^2 = dx^2 + dy^2 \tag{11}$$

and $\bar{\mu} = 1$.

Defining the complex variables $z = (x, y)$ and $\zeta = (\psi, \omega)$, a convenient general expression for the conformal projection of the spheroid on the mapping plane in the form of (4) is

$$z = p_0 f(\zeta) \tag{12}$$

where p_0 is the radius of the parallel of latitude $\phi = \phi_0$. From (5) the scale factor m is given by

$$m = \frac{1}{p}\left|\frac{\partial z}{\partial \zeta}\right| = \frac{p_0}{p}\left|\frac{\partial f(\zeta)}{\partial \zeta}\right| = \frac{p_0}{p}|\sigma| \tag{13}$$

where

$$\sigma = \frac{\partial f(\zeta)}{\partial \zeta} \tag{14}$$

and the meridian convergence γ is given by

$$\gamma = \arg(\sigma) \tag{15}$$

The transformation function $f(\zeta)$ can be expressed in a most general form by an infinite power series

$$f(\zeta) = \sum_{n=0}^{\infty} B_n \zeta^n \tag{16}$$

where the complex coefficients B_n are to be found to satisfy conditions for origin, orientation and scale. In practice we must truncate the series after $N+1$ terms, say, thus expressing the transformation by the polynomial

$$z = p_0 \sum_{n=0}^{N} B_n \zeta^n \tag{17}$$

with scale factor $m = |\sigma|\, p_0/p$, where

$$\sigma = \frac{\partial f(\zeta)}{\partial \zeta} = \sum_{n=1}^{N} n B_n \zeta^{n-1} = \sum_{n=1}^{N} B_n Z_n \tag{18}$$

where

$$Z_n = n\zeta^{n-1}.$$

Equation (17) expresses a direct transformation of the isometric coordinates ζ on the spheroid into the rectangular coordinates z on the mapping plane, but the process may be interpreted in two stages as

(*a*) the transformation of the isometric coordinates ζ on the spheroid into rectangular Mercator projection coordinates ξ on the mapping plane by

$$\xi = p_0 \zeta \tag{19}$$

with an associated scale factor m_1 given by

$$m_1 = \frac{1}{p} \left| \frac{\partial \xi}{\partial \zeta} \right| = p_0/p \tag{20}$$

followed by

(*b*) the transformation of the Mercator projection coordinates ξ into the desired rectangular coordinates z by

$$\left.\begin{aligned} z &= p_0 \sum_{n=0}^{N} B_n (\xi/p_0)^n \\ &= \sum_{n=0}^{N} C_n \xi^n \end{aligned}\right\} \tag{21}$$

where $C_n = p_0{}^{1-n} B_n$, and the associated scale factor m_2 is given by

$$\left.\begin{aligned} m_2 &= \left| \frac{\partial z}{\partial \xi} \right| = \left| \sum_{n=1}^{N} n C_n \xi^{n-1} \right| \\ &= \left| \sum_{n=1}^{N} n B_n \zeta^{n-1} \right| = |\sigma| \end{aligned}\right\} \tag{22}$$

The overall scale factor m is

$$m = m_1 m_2 = |\sigma|\, p_0/p \tag{23}$$

Determination of the Coefficients to give a Minimum Scale Error Projection

In the general expression (17) for transforming the geodetic coordinates ζ into the rectangular coordinates z, we must now assign values to the $N+1$ complex coefficients B_n (a total of $2N+2$ parameters).

Firstly, the coordinate z can be given a desired value $z_0 = (x_0, y_0)$ at the origin $\zeta = 0$, whence

$$B_0 = z_0/p_0 \tag{24}$$

Secondly, the angle between the x axis (grid north) and the ψ axis (true north) can be given a desired value γ_0 at the origin, i.e.

$$\arg(\sigma) = \arg(B_1) = \gamma_0 \tag{25}$$

and if the grid north and true north are to coincide at the origin, then $\gamma_0 = 0$, $\arg(B_1) = 0$, and thus $\mathrm{Im}(B_1) = 0$. Assuming the latter case, we have now assigned 3 parameters, and have $2N-1$ remaining to satisfy a scale condition.

By analogy with the "balance of errors" projection of Airy **[1]**, we may define the total distortion T^2 of the projection over an arbitrary area S on the spheroid by

$$T^2 = \frac{1}{A}\iint_S k^2\,dA = \frac{1}{A}\iint_S (m-1)^2\,dA \tag{26}$$

where A is the total area of S given by

$$A = \iint_S dA \tag{27}$$

An irregular area (the land area of New Zealand, for example), can be subdivided into M regular elements of extent $\Delta\phi$ in latitude and $\Delta\lambda$ in longitude, and by substituting $dA = \cos\phi\,d\phi\,d\lambda$ by spherical approximation, the continuous integral form of (26) can be expressed as the finite sum

$$T^2 = \sum_{i=1}^{M} \cos\phi_i\{m(\phi_i, \lambda_i) - 1\}^2 \Big/ \sum_{i=1}^{M} \cos\phi_i \tag{28}$$

where the (ϕ_i, λ_i) are the geodetic coordinates of the centres of each element of area.

The remaining $2N-1$ parameters in the polynomial expression (17) of the conformal transformation will therefore be chosen in such a way as to minimise the total distortion T^2 given by (28). To do this, it is first necessary to express the scale factor $m = |\sigma|\,p_0/p$ as a linear function of unknowns, since $|\sigma|$ is not a linear function of the unknown B_n. The solution can be achieved by a method of successive approximation.

Let $B_n{}^*$ be a set of trial coefficients, and ΔB_n a set of small correction terms to be found to yield the corrected estimate

$$B_n = B_n{}^* + \Delta B_n. \tag{29}$$

If $\sigma^* = \sum_{n=1}^{N} B_n{}^* Z_n$ is the trial value of σ, and $\Delta\sigma = \sum_{n=1}^{N} \Delta B_n Z_n$, then $|\sigma|$ may be expressed as

$$\left.\begin{aligned}|\sigma| &= [\sigma\tilde{\sigma}]^{\frac{1}{2}} \\ &= [(\sigma^*+\Delta\sigma)(\tilde{\sigma}^*+\Delta\tilde{\sigma})]^{\frac{1}{2}} \\ &= [\sigma^*\,\tilde{\sigma}^*+\sigma^*\,\Delta\tilde{\sigma}+\tilde{\sigma}^*\,\Delta\sigma+\Delta\sigma\,\Delta\tilde{\sigma}]^{\frac{1}{2}}\end{aligned}\right\} \tag{30}$$

Neglecting $\Delta\sigma\,\Delta\tilde{\sigma}$ by comparison with the other terms, and noting that

$$\sigma^*\,\Delta\tilde{\sigma}+\tilde{\sigma}^*\,\Delta\sigma = 2\,\mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma}),$$

we have

$$\left.\begin{aligned}|\sigma| &\approx [\sigma^*\,\tilde{\sigma}^*+2\mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma})]^{\frac{1}{2}} \\ &\approx (\sigma^*\,\tilde{\sigma}^*)^{\frac{1}{2}}\left[1+\frac{\mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma})}{\sigma^*\,\tilde{\sigma}^*}\right]\end{aligned}\right\} \tag{31}$$

from the first two terms of the binomial expansion. Since $(\sigma^*\,\tilde{\sigma}^*)^{\frac{1}{2}} = |\sigma^*|$, (31) may be written

$$|\sigma| = |\sigma^*|+\frac{\mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma})}{|\sigma^*|} \tag{32}$$

Expanding $\Delta\tilde{\sigma}$ as $\sum_{n=1}^{N}\Delta\tilde{B}_n\,\tilde{Z}_n$ and writing H for $\mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma})$ it follows that

$$H = \mathrm{Re}\,(\sigma^*\,\Delta\tilde{\sigma}) = \sum_{n=1}^{N}\{\mathrm{Re}\,(\Delta B_n)\,\mathrm{Re}\,(\sigma^*\,\tilde{Z}_n)+\mathrm{Im}\,(\Delta B_n)\,\mathrm{Im}\,(\sigma^*\,\tilde{Z}_n)\} \tag{33}$$

which is a linear function of the unknowns $\mathrm{Re}\,(\Delta B_n)$ and $\mathrm{Im}\,(\Delta B_n)$. Hence

$$m-1 = (m^*-1)+\left(\frac{p_0}{p}\right)\frac{H}{|\sigma^*|} \tag{34}$$

and substituting this in (28) we have

$$T^2 = \sum_{i=1}^{M}\cos\phi_i\left\{(m^*-1)+\left(\frac{p_0}{p}\right)\frac{H}{|\sigma^*|}\right\}^2\Big/\sum_{i=1}^{M}\cos\phi_i \tag{35}$$

Introducing the symbol b_k for the unknowns $\mathrm{Re}\,(\Delta B_k)$ and $\mathrm{Im}\,(\Delta B_k)$ we may now differentiate T^2 with respect to all the $2N-1$ possible values of b_k and equate the results to zero, leading to $2N-1$ equations of the form

$$\sum_{i=1}^{M}\cos\phi_i\left\{(m^*-1)+\left(\frac{p_0}{p}\right)\frac{H}{|\sigma^*|}\right\}\left(\frac{p_0}{p}\right)\frac{1}{|\sigma^*|}\frac{\partial H}{\partial b_k} = 0 \tag{36}$$

or rearranging terms

$$\sum_{i=1}^{M}\cos\phi_i\left(\frac{p_0}{p}\right)^2\frac{H}{|\sigma^*|^2}\frac{\partial H}{\partial b_k} = \sum_{i=1}^{M}\cos\phi_i\left(\frac{p_0}{p}\right)\frac{(1-m^*)}{|\sigma^*|}\frac{\partial H}{\partial b_k} \tag{37}$$

which, by substituting for H from (33) may be recast in the form

$$\sum_{n=1}^{N}\alpha_{nk}\,\mathrm{Re}\,(\Delta B_n)+\sum_{n=2}^{N}\beta_{nk}\,\mathrm{Im}\,(\Delta B_n) = \delta_k \tag{38}$$

where

$$\alpha_{nk} = \sum_{i=1}^{M} \cos\phi_i \left(\frac{p_0}{p}\right)^2 \frac{\mathrm{Re}\,(\sigma^* \tilde{Z}_n)}{|\sigma^*|^2} \frac{\partial H}{\partial b_k} \tag{39}$$

$$\beta_{nk} = \sum_{i=1}^{M} \cos\phi_i \left(\frac{p_0}{p}\right)^2 \frac{\mathrm{Im}\,(\sigma^* \tilde{Z}_n)}{|\sigma^*|^2} \frac{\partial H}{\partial b_k} \tag{40}$$

$$\delta_k = \sum_{i=1}^{M} \cos\phi_i \left(\frac{p_0}{p}\right) \frac{(1-m^*)}{|\sigma^*|} \frac{\partial H}{\partial b_k} \tag{41}$$

and $\dfrac{\partial H}{\partial b_k}$ takes the value

$$\left.\begin{aligned} \frac{\partial H}{\partial b_k} &= \mathrm{Re}\,(\sigma^* \tilde{Z}_k) \quad (k = 1, 2, 3 \ldots N) \text{ for } b_k = \mathrm{Re}\,(\Delta B_k) \\ &\ \mathrm{Im}\,(\sigma^* \tilde{Z}_k) \quad (k = 2, 3, \ldots N) \text{ for } b_k = \mathrm{Im}\,(\Delta B_k) \end{aligned}\right\} \tag{42}$$

The set of $2N-1$ simultaneous equations of the form (38) can thus be solved to give the unknown correction terms ΔB_n. The corrected terms (29) can then be used as trial values in the next iteration, and the process repeated until the corrections become sufficiently small.

Since $m \approx 1$ over the area of the projection, then $|\sigma| \approx p/p_0$, a function of ψ only, and thus

$$\frac{\partial|\sigma|}{\partial\psi} \approx \frac{1}{p_0}\frac{\partial p}{\partial\psi} = \frac{p}{p_0}\frac{\partial \ln p}{\partial\psi} = -\frac{p}{p_0}\sin\phi, \text{ and } \frac{\partial|\sigma|}{\partial\omega} \approx 0$$

Taking values at the origin $\phi = \phi_0$, a convenient first approximation to the trial values B_n^* is found to be $\mathrm{Re}\,(B_1) = 1$, $\mathrm{Re}\,(B_2) = -\frac{1}{2}\sin\phi_0$, with the remaining terms zero.

Further Properties of the Projection

Since from (13) and (15), $|\sigma| = mp/p_0$ and $\arg(\sigma) = \gamma$, we may write

$$\ln\sigma = (\ln mp/p_0, \gamma) \tag{43}$$

Differentiating this expression with respect to ζ,

$$\left.\begin{aligned} \frac{\partial \ln\sigma}{\partial\zeta} &= \left(\frac{\partial}{\partial\psi}, -\frac{\partial}{\partial\omega}\right) \ln mp/p_0 = \left(\frac{\partial \ln m}{\partial\psi} - \sin\phi, -\frac{\partial \ln m}{\partial\omega}\right) \\ &= \left(\frac{\partial\gamma}{\partial\omega}, \frac{\partial\gamma}{\partial\psi}\right) \end{aligned}\right\} \tag{44}$$

it follows that the gradients of the scale factor m and the meridian convergence γ with respect to the isometric coordinates (ψ, ω) are given by

$$\left(\frac{\partial \ln m}{\partial\psi}, -\frac{\partial \ln m}{\partial\omega}\right) = \frac{\partial \ln\sigma}{\partial\zeta} + \sin\phi\ . \tag{45}$$

and

$$\left(\frac{\partial \gamma}{\partial \omega}, \frac{\partial \gamma}{\partial \psi}\right) = \frac{\partial \ln \sigma}{\partial \zeta} \tag{46}$$

Since $\dfrac{\partial}{\partial z} = \dfrac{1}{p_0 \sigma} \dfrac{\partial}{\partial \zeta}$, the corresponding gradients with respect to the rectangular coordinates (x, y) are given by

$$\left(\frac{\partial \ln m}{\partial x}, -\frac{\partial \ln m}{\partial y}\right) = \frac{1}{p_0 \sigma}\left(\frac{\partial \ln \sigma}{\partial \zeta} + \sin \phi\right) \tag{47}$$

and

$$\left(\frac{\partial \gamma}{\partial y}, \frac{\partial \gamma}{\partial x}\right) = \frac{1}{p_0 \sigma} \frac{\partial \ln \sigma}{\partial \zeta} \tag{48}$$

The curvatures of lines projected from the spheroid onto the mapping plane may be examined by the use of Schols' formula (Näbauer, [8])

$$\bar{G} = \frac{1}{m} G - \frac{\partial \ln m}{\partial \bar{n}} \tag{49}$$

where G is the geodesic curvature of a curve on the spheroid at a point P, $\bar{G}$ the geodesic curvature of its projection on the plane at the corresponding point $\bar{P}$, and $\bar{n}$ the vector normal to the curve at $\bar{P}$ in the mapping plane (Fig. 2). If G_m is the geodesic curvature of a curve C_m through P on the spheroid, whose tangent at P makes an angle θ with the meridian through P, and G_p is the geodesic curvature of the conjugate curve C_p normal to C_m at P, then we may define the complex curvature $\Gamma = (G_p, G_m)$ and the unit vector $\Theta = (\cos \theta, \sin \theta)$. The curvature of this pair of projected curves is then given by

$$\left.\begin{aligned} \bar{\Gamma} &= \frac{\sigma \Theta}{|\sigma|} \frac{\partial \ln (\sigma/p)}{\partial z} + \frac{1}{m} \Gamma \\ &= \frac{1}{m}\left\{\frac{\Theta}{p}\left[\frac{\partial \ln \sigma}{\partial \zeta} + \sin \phi\right] + \Gamma\right\} \end{aligned}\right\} \tag{50}$$

If the conjugate curves on the spheroid are both geodesics, then their geodesic curvatures are zero, and $\Gamma = 0$ in (50). To examine the curvatures of meridians and parallels in the projection, we shall take the curve C_p on the spheroid to be a parallel, whose geodesic curvature is $G_p = -\sin \phi / p$, a real quantity. The angle θ is zero, whence $\Theta = (1, 0)$, the identity element for complex multiplication. Substituting for Γ and Θ in (50), we have for the curvature of parallels and meridians in the projection

$$\bar{\Gamma} = \frac{1}{mp} \frac{\partial \ln \sigma}{\partial \zeta} \tag{51}$$

The general expressions for the curvatures of projected geodesics (50) can be used in the calculation of the corrections to the bearings of field measurements for calculations made in terms of the rectangular coordinates z in the mapping plane.

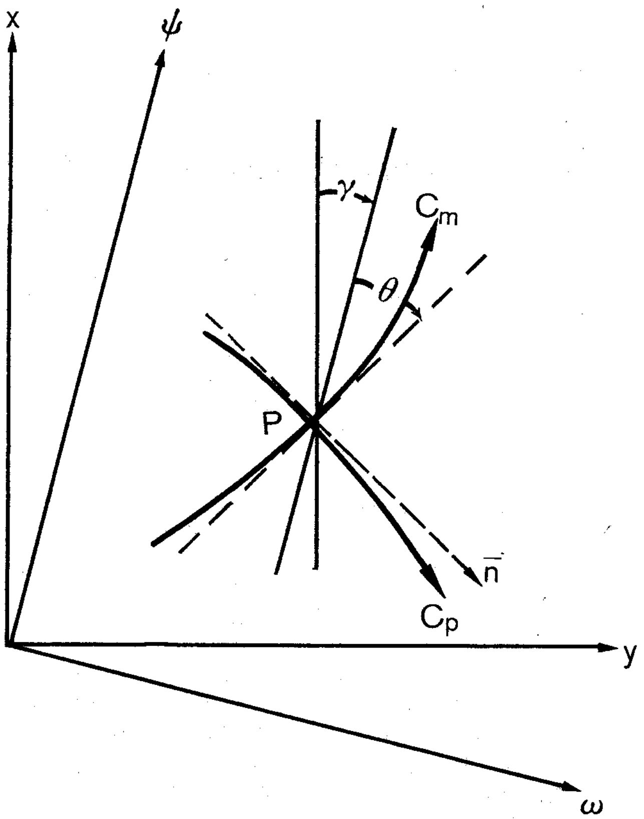

Fig. 2. The geodesic curvature of projected curves. (x, y) are the rectangular coordinates, and (ψ, ω) the isometric coordinates of the spheroid. The meridian convergence at P is γ. The curve C_m through P makes an angle θ with the meridian (ω = const) and has a geodesic curvature G_m; C_p is a curve orthogonal to C_m at P, and has a geodesic curvature G_p. The direction $\bar{n}$ is normal to the curve C_m at P.

A geodesic from A to B on the spheroid will project as a curve on the mapping plane (Fig. 3). If the bearing of the tangent to the projected geodesic at A is α, and the bearing of the chord AB is β, then the arc-to-chord correction δ is

$$\delta = \beta - \alpha \qquad (52)$$

On the assumption that the form of the projected geodesic can be approximated by a cubic parabola, Hotine **[4]** gives a value of δ as

$$\delta = \tfrac{1}{2}SG^* \qquad (53)$$

where S is the chord length AB, and G^* is the curvature of the geodesic at a point one-third of the distance from A to B along the chord. The curvature G^* can be calculated from (50) as $G^* = \mathrm{Im}(\bar{\Gamma})$ by putting $\Gamma = 0$ and $\theta = \beta - \gamma$. Defining $\Upsilon = (\cos\beta, \sin\beta)$, we have

$$G^* = \mathrm{Im}(\bar{\Gamma}) = \mathrm{Im}\left\{\frac{\Upsilon}{p_0\,\sigma}\left[\frac{\partial \ln \sigma}{\partial \zeta} + \sin\phi\right]\right\} \qquad (54)$$

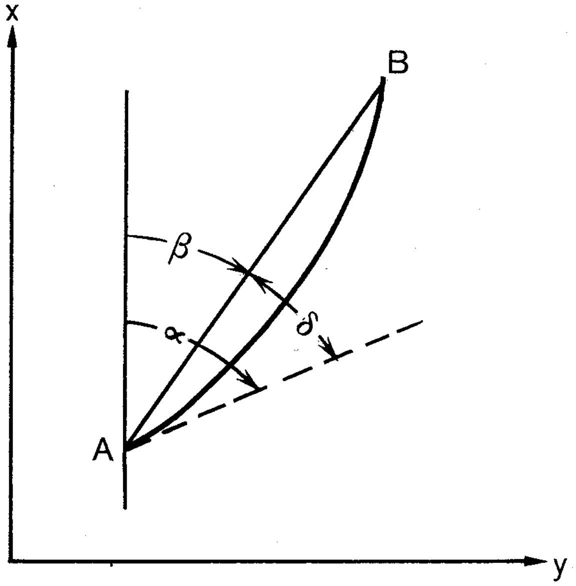

Fig. 3. The arc-to-chord correction δ in the projection for the geodesic AB. α is the bearing of tangent to the projected geodesic at A, and β the bearing of the chord AB.

A Minimum Scale Error Projection for New Zealand

Tests were made to design a minimum scale error projection to map the land area of New Zealand. This area was represented by a total of 228 points chosen at half-degree intervals of latitude and longitude. The origin of the projection was taken at $\phi_0 = 41°S$, $\lambda_0 = 173°E$ with the convergence γ_0 set at zero at the origin. Coefficients were calculated for transformation polynomials of order N ranging from 4 to 12. Beyond $N = 6$ there was little reduction in the total distortion T^2, and the curves of equal scale error became increasingly complex, so we have chosen the case for $N = 6$ to illustrate the properties of the projection. The total distortion (i.e. the root-mean-square scale error) for this order was $1{\cdot}2 \times 10^{-4}$, and the curves of equal scale error are shown in Fig. 4 in units of 1×10^{-4} (100 mm/km). This can be compared with the scale error curves for the existing transverse Mercator projections of the country (Fig. 1).

It is seen that for the transverse Mercator projection, the range of scale error is about 20×10^{-4} in the South Island, and about 8×10^{-4} in the North Island, whereas for the minimum scale error projection the range of scale error is about 4×10^{-4}. The projection may also be compared with the oblique conical projection (of the sphere, not of the spheroid) devised by Craster [2] for New Zealand, which has a range of scale error of about 12×10^{-4}.

Also shown in Fig. 5 are contours of the maximum possible value of the arc-to-chord correction at any point, in units of $0{\cdot}1''$/km, using Hotine's expression (53) for the arc-to-chord correction. This reaches its greatest value in the northernmost part of the country, which forms a salient at right angles to the main trend of the land area, and is the most difficult part to accommodate within the minimum scale error projection. The graticule is not shown plotted in the chosen projection, for at the scale of reproduction it would be indistinguishable from that of a Transverse Mercator projection, for example.

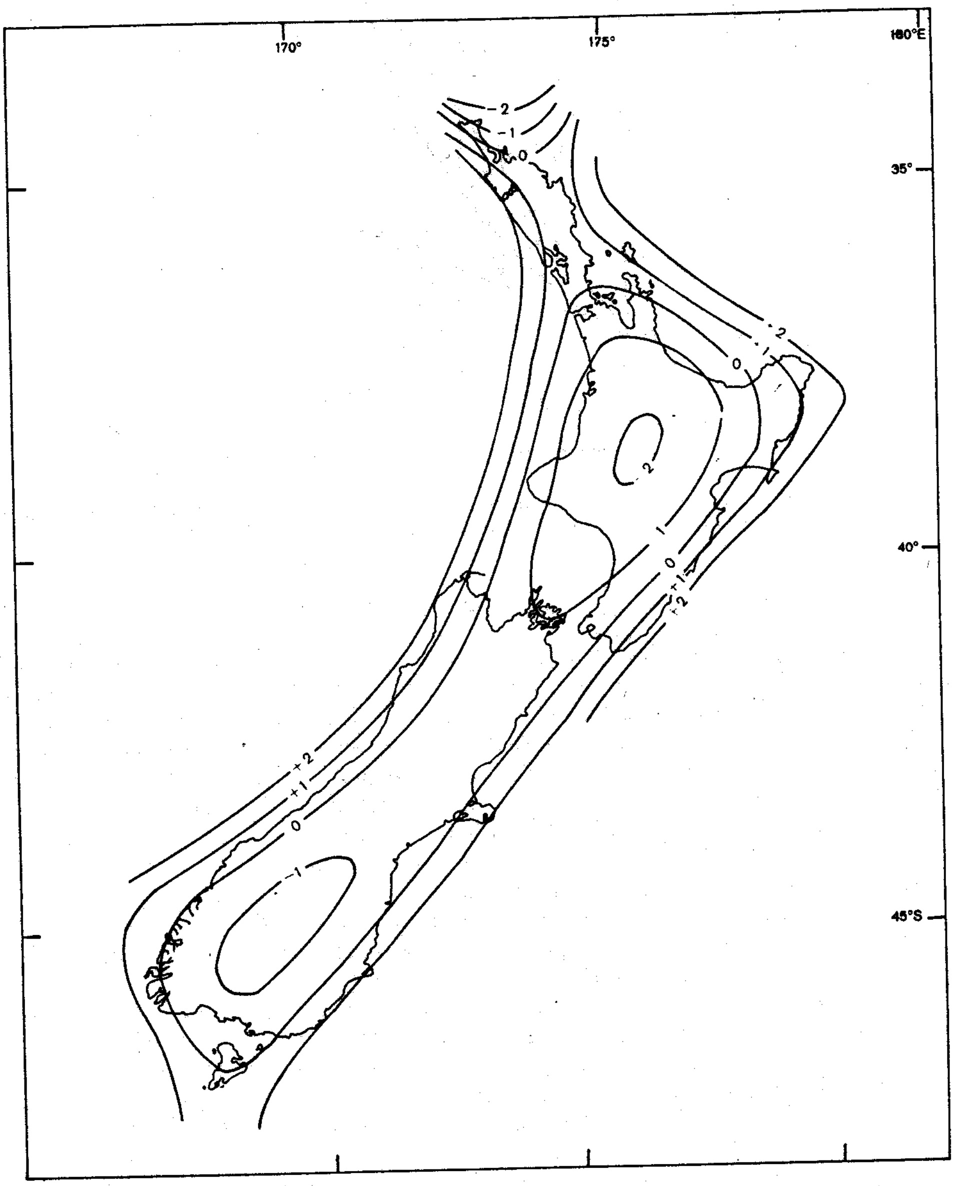

Fig. 4. Curves of equal scale error of a minimum scale error conformal projection for New Zealand, using a transformation expressed as a complex polynomial of order six. Scale error in units of 10^{-4} (100 mm/km).

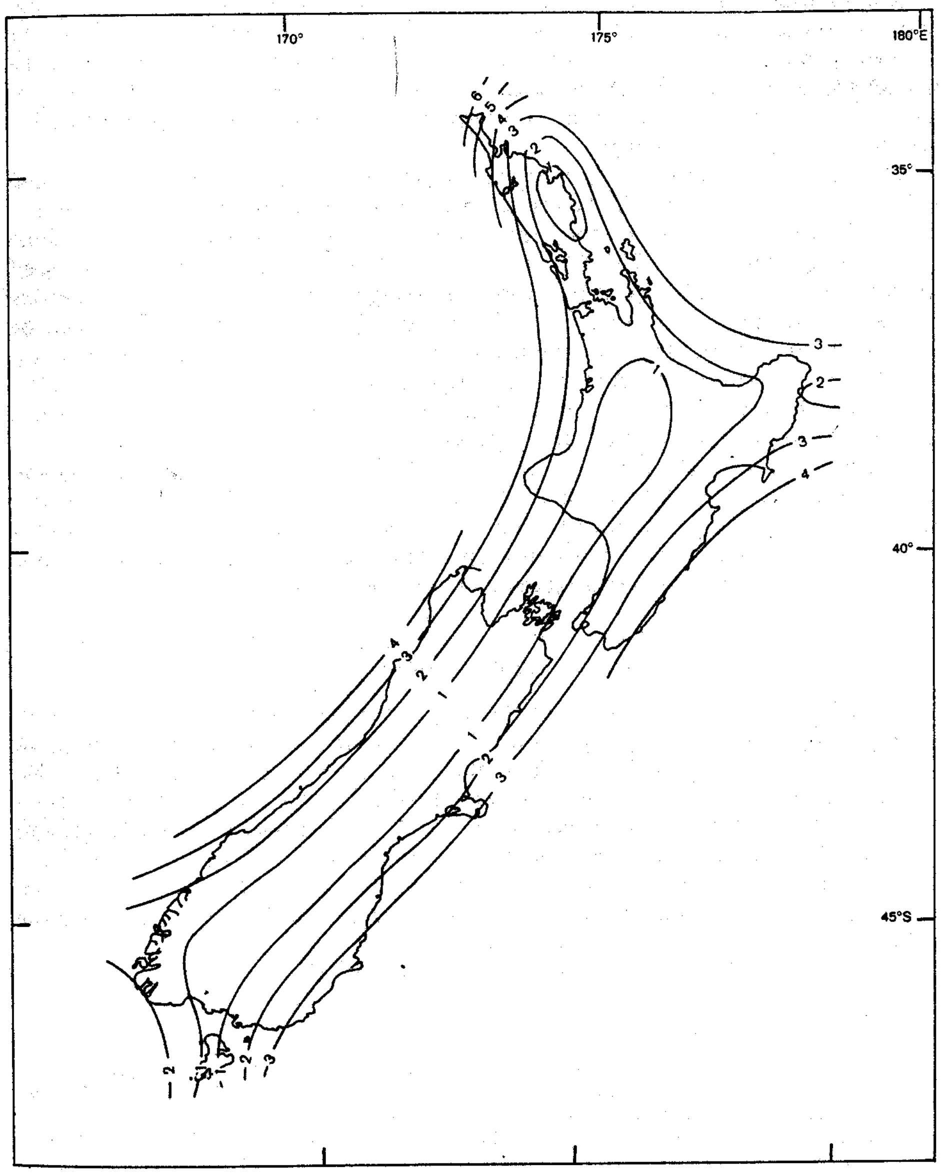

Fig. 5. Curves of equal maximum arc-to-chord correction for the minimum scale error projection of Fig. 4. Maximum correction in units of 0·1″/km.

CONCLUSIONS

This investigation has sought to demonstrate that by adopting a most general mathematical form of conformal transformation, and discarding the restriction of projecting on regular developable surfaces, it is possible to devise a conformal projection which has a range of scale error markedly less than that of any conventional projection. The advantages of this type of projection lie mainly with the surveyor, who can reduce local surveys in the rectangular coordinate system of the projection, either neglecting the small corrections to bearing and distance, or determining them by rapid graphical interpolation from contour maps prepared in advance for the whole country.

The various calculations required are certainly no more complicated than those required for the Transverse Mercator projection, and the preparation of the necessary tables and maps by the central survey organisation can be readily done by electronic computer. The set of $N+1$ complex coefficients B_n can be used directly in all the required formulae, thus obviating the need to calculate numerous derived coefficients. Transformation from geodetic to isometric latitude can be done by a series formula with coefficients appropriate to the origin latitude ϕ_0. Transformation from isometric to rectangular coordinates is done directly by equation (17), and the inverse transformation can be done most easily by Newton's method of inverse interpolation using the same formula.

The minimum scale error projection described above is suited to the mapping of isolated areas, such as New Zealand, where no " rolling fit " to adjoining projections is required. If a conformal projection with minimum scale error over a given area is desired, then one designed specifically to meet this need has obvious advantages over conventional projections not so designed.

ACKNOWLEDGMENTS

I am indebted to Mr L. P. Lee, Chief Computer, Lands and Survey Department, Wellington, for his original suggestion of the problem, for provision of a number of useful formulae, and for many valuable discussions of the subject. I am also grateful for discussions of the problem with Professor J. B. Mackie, University of Otago, and Mr J. Holmes Miller, Wellington, and for the advice and encouragement given by Dr T. Hatherton and Dr M. P. Hochstein of Geophysics Division, D.S.I.R. Finally, I am indebted to Dr H. M. Bibby of Geophysics Division, D.S.I.R., for much valuable assistance with the mathematical aspects of the problem.

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Appendix: Notation for Complex Numbers

A complex number u can be defined as an ordered pair of real numbers a, b

$$u = (a, b)$$

where $u = 0$ if both $a = 0$ and $b = 0$.

The " real " part of u is $\mathrm{Re}(u) = a$, and the " imaginary " part is $\mathrm{Im}(u) = b$. Two complex numbers $u = (a, b)$ and $v = (c, d)$ are equal only if $a = c$ and $b = d$. The identity element for addition is $(0, 0)$, and for multiplication is $(1, 0)$. The inverse of $u = (a, b)$ for addition is $-u = (-a, -b)$, and for multiplication is

$$\frac{1}{u} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right).$$

Arithmetic operations with complex numbers are interpreted as follows:

addition	$u+v$	$= (a+c, b+d)$
subtraction	$u-v$	$= (a-c, b-d)$
multiplication	uv	$= (ac-bd, ad+bc)$
division	u/v	$= \left(\frac{ac+bd}{c^2+d^2}, \frac{-ad+bc}{c^2+d^2}\right)$

The complex conjugate of $u = (a, b)$ is $\tilde{u} = (a, -b)$, whence $u\tilde{u} = (a^2+b^2, 0)$. The modulus of $u = (a, b)$ is $|u| = \sqrt{(a^2+b^2)} = (u\tilde{u})^{\frac{1}{2}}$. The argument of $u = (a, b)$ is $\arg(u) = \arctan(b/a)$. If $u = f(z)$, where $u = (a, b)$ and $z = (x, y)$, and a and b are functions of x and y, then

$$\frac{\partial u}{\partial z} = \left(\frac{\partial a}{\partial x}, \frac{\partial b}{\partial x}\right) = \left(\frac{\partial b}{\partial y}, -\frac{\partial a}{\partial y}\right)$$

and the Cauchy Riemann differential equations are

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}, \frac{\partial b}{\partial x} = -\frac{\partial a}{\partial y}$$